DETERMINING THE WEAK PHASE γ
FROM CHARGED B DECAYS

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ABSTRACT

A quadrangle relation is shown to be satisfied by the amplitudes for $B^+ \to \pi^0 K^+$, $\pi^+ K^0$, ηK^+, and $\eta' K^+$. By comparison with the corresponding relation satisfied by B^- decay amplitudes, it is shown that the relative phases of all the amplitudes can be determined up to discrete ambiguities. Making use of an SU(3) relation between amplitudes contributing to the above decays and those contributing to $B^\pm \to \pi^\pm \pi^0$, it is then shown that one can determine the weak phase $\gamma \equiv \text{Arg}(V_{ub}^* V_{cb}/V_{us}^* V_{cs})$, where V is the Cabibbo-Kobayashi-Maskawa matrix describing the charge-changing weak interactions between the quarks (u, c, t) and (d, s, b).

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I. INTRODUCTION

The presence of phases in elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix [1] is the currently favored explanation of the observed violation of CP invariance [2] in the neutral kaon system. However, independent tests for the presence of these phases are needed. One class of such tests is based on observing the predicted violation of CP invariance in B meson decays. In order to interpret such violations in terms of CKM phases one needs to identify the flavor of the decaying B meson, posing potential "tagging" problems for neutral B's, or to understand final-state interactions in "self-tagging" modes such as $B^{\pm} \to \pi K$.

Using flavor SU(3) [3–6] we proposed [7–9] that $B^{\pm} \to \pi\pi$ and $B^{\pm} \to \pi K$ amplitudes could be related to one another in such a way as to obtain the weak phase $\gamma \equiv \mathrm{Arg}(V_{ub}^* V_{cb}/V_{us}^* V_{cs})$, where V_{ij} is the element of the CKM matrix describing the charge-changing weak interaction between the quarks $i = (u, c, t)$ and $j = (d, s, b)$. It was shown [10, 11] that electroweak penguin contributions [12] could spoil this relation, and were likely to be significant. We proposed a way to specify the phase γ independently of these contributions but making use of a rare decay $B_s \to \pi^0 \eta$ [13]. A simpler relation yields similar information by employing the decays $B^{\pm} \to \eta_8 K^{\pm}$, where η_8 denotes an octet member. The amplitude triangle relation [14]

$$A(B^+ \to \pi^0 K^+) + \sqrt{2}A(B^+ \to \pi^+ K^0) = \sqrt{3}A(B^+ \to \eta_8 K^+) \tag{1}$$

(also implied by the first three lines of Table II of Ref. [7]) can be compared with the relation for the charge-conjugate process to yield an amplitude which, when normalized by that governing $B^{\pm} \to \pi^{\pm}\pi^0$, yields the phase γ.

A modified version of the triangle relation (1) can be applied to the physical η, an octet-singlet mixture, only if one neglects the independent amplitudes in which the singlet component of the η is produced. In the present paper we identify the most important of these amplitudes and show how to recover information on γ in its presence.

We shall show that the natural generalization of (1) for an octet-singlet mixture, the quadrangle relation

$$A(B^+ \to \pi^0 K^+) + \sqrt{2}A(B^+ \to \pi^+ K^0) = \sqrt{3}[\cos\theta A(B^+ \to \eta K^+) + \sin\theta A(B^+ \to \eta' K^+)] , \tag{2}$$

where

$$\eta \equiv \eta_8 \cos\theta - \eta_1 \sin\theta \quad , \quad \eta' \equiv \eta_8 \sin\theta + \eta_1 \cos\theta \quad , \tag{3}$$

and $\eta_8 \equiv (2s\bar{s} - u\bar{u} - d\bar{d})/\sqrt{6}$, $\eta_1 \equiv (s\bar{s} + u\bar{u} + d\bar{d})/\sqrt{3}$, can be compared with the corresponding relation for B^- decays in order to learn the shape of both quadrangles. One can then form a difference between two amplitudes for $B \to \pi K$ and $\bar{B} \to \pi\bar{K}$ which, when compared with the amplitude for $B^{\pm} \to \pi^{\pm}\pi^0$, provides the weak phase γ.

We introduce notation and assumptions and describe decay amplitudes in terms of four independent quantities in Sec. II. The quadrangle (2) and that for B^- decays are constructed in Sec. III, where we also obtain an expression for γ. Some comments regarding SU(3) breaking and experimental considerations occupy Sec. IV, while Sec. V

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concludes. An explicit geometric construction of amplitude quadrangles is described in an Appendix.

II. NOTATION, ASSUMPTIONS AND AMPLITUDES

A. Definition of states

For SU(3) amplitudes, we adopt a graphical notation described in more detail in Refs. [7, 8, 9, 13, 15]. Our states are defined by

$$\pi^+ = u\bar{d} \quad , \quad \pi^0 = (d\bar{d} - u\bar{u})/\sqrt{2} \quad , \quad \pi^- = -d\bar{u} \quad ,$$

$$K^+ = u\bar{s} \quad , \quad K^0 = d\bar{s} \quad , \quad \bar{K}^0 = s\bar{d} \quad , \quad K^- = -s\bar{u} \quad ,$$

$$\eta_8 \equiv (2s\bar{s} - u\bar{u} - d\bar{d})/\sqrt{6} \quad , \quad \eta_1 \equiv (s\bar{s} + u\bar{u} + d\bar{d})/\sqrt{3} \quad , \tag{4}$$

with (3) describing the physical η and η'. A good approximation, corresponding to an octet-singlet mixing angle [5, 16] $\theta = \theta_p \equiv \arcsin{(1/3)} \simeq 19.5°$, is the representation

$$\eta = \eta_p \equiv \frac{s\bar{s} - u\bar{u} - d\bar{d}}{\sqrt{3}} \quad , \quad \eta' = \eta'_p \equiv \frac{2s\bar{s} + u\bar{u} + d\bar{d}}{\sqrt{6}} \quad . \tag{5}$$

B. Assumptions about amplitudes

The decays $B \to M_8 M_8$, where M_8 are pseudoscalar mesons belonging to octets of flavor SU(3), are characterized by 5 independent amplitudes, corresponding to one **27**, three **8**'s, and one **1** in the direct channel [3]. (We denote a flavor SU(3) representation by its dimension in bold face.) In previous works [7–9, 13, 15] we have argued that the neglect of amplitudes containing factors of f_B/m_B is equivalent to relations between the **27** and one of the **8**'s, and between the **1** and another of the **8**'s, leaving 3 independent amplitudes. These can be characterized by graphs T, C, P, illustrated in Fig. 1, in which the spectator quark in the decaying B does not enter into the decay Hamiltonian. A number of tests were proposed [7] for the description of B decays in terms of this restricted set of SU(3) amplitudes.

The presence of electroweak penguin contributions [10–13] does not alter the validity of an SU(3) description, as long as one relates amplitudes with the same strangeness change ($|\Delta S| = 0$ or 1) to one another. In that case one may simply substitute

$$T \to t \equiv T + P_{EW}^C \quad , \tag{6}$$

$$C \to c \equiv C + P_{EW} \quad , \tag{7}$$

$$P \to p \equiv P - \frac{1}{3}P_{EW}^C \quad , \tag{8}$$

where P_{EW} and P_{EW}^C, the color-favored and color-suppressed electroweak penguin amplitudes, correspond to the graphs in Fig. 2.
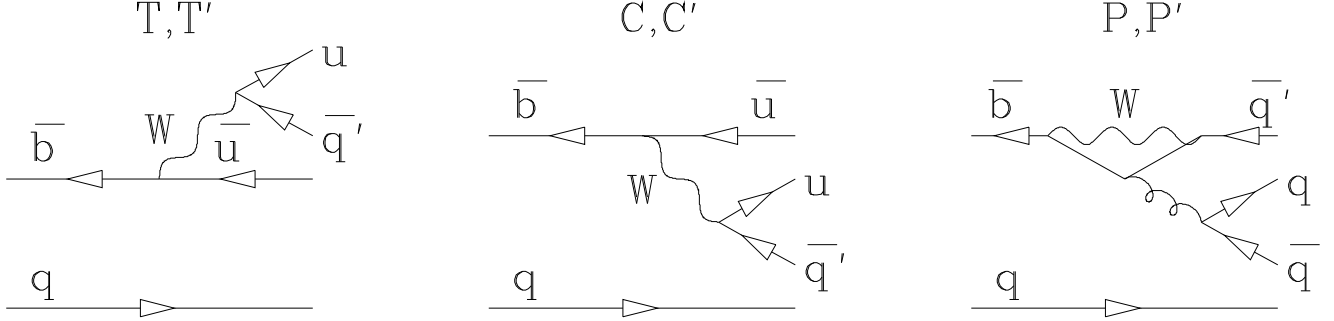
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Figure 1: Graphs contributing to $B \to M_8 M_8$ decays which are not suppressed by a factor of f_B/m_B. Here $q = u, d, s$, while $q' = d$ for unprimed amplitudes and s for primed amplitudes. The coiled line in the third graph denotes exchange of one or more gluons.
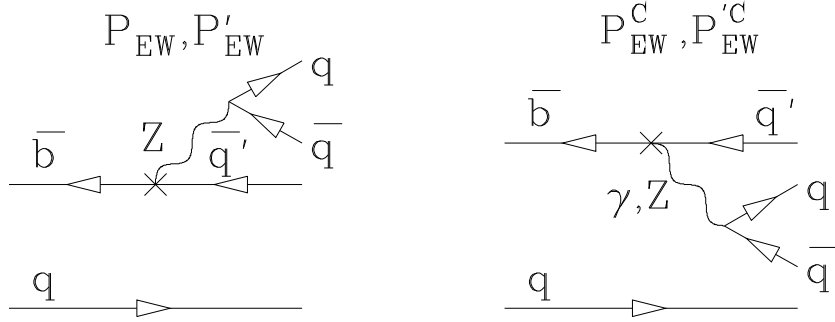

Figure 2: Electroweak penguin graphs contributing to $B \to M_8 M_8$ decays. The crosses denote loops involving W exchange. Not shown are additional WW box diagrams required for gauge invariance.
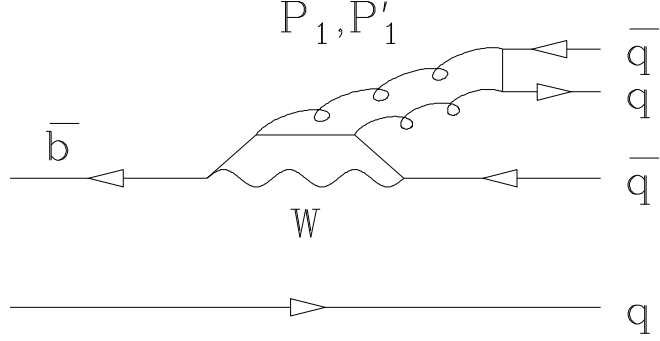
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Figure 3: Graph contributing to $B \to M_1 M_8$ and $B \to M_1 M_1$. The coiled lines denote a color singlet exchange due to two or more gluons.

When one or two singlet pseudoscalar mesons M_1 are in the final state, additional amplitudes must be taken into account [3, 17]. For the decays $B \to M_1 M_8$, since the final state is an octet, there are three **8** amplitudes corresponding to the three different representations in the weak Hamiltonian H_W describing $\bar{b} \to \bar{q}_1 q_2 \bar{q}_3$, where q_i are light (u, d, s) quarks. These representations transform as $\mathbf{3}^*$, **6**, or $\mathbf{15}^*$ of SU(3). When combined with the **3** of the spectator quark, each contains one octet. For the decays $B \to M_1 M_1$, there is one singlet obtained from the product of the $\mathbf{3}^*$ in H_W and the **3** of the spectator quark.

As in the case of $M_8 M_8$ production, we now neglect all amplitudes in which the spectator quark enters into the decay Hamiltonian. We thus identify a single new amplitude, depicted in Fig. 3, contributing to $M_1 M_8$ and $M_1 M_1$ production. This amplitude is denoted by P_1. This assumption has also been adopted in Ref. [17], where many tests of it are proposed.

An additional electroweak penguin contribution occurs whenever one has $M_1 M_8$ or $M_1 M_1$ production. Since it always appears in a fixed combination with respect to P_1, we have one further amplitude

$$p_1 \equiv P_1 - \frac{1}{3} P_{EW} \quad , \tag{9}$$

in addition to (6)–(8), describing decays involving singlets.

C. Summary of amplitudes

We shall denote amplitudes corresponding to $\Delta S = 0$ without primes and those for $|\Delta S| = 1$ with primes. The amplitudes of interest for $|\Delta S| = 1$ decays are expressed in terms of the four independent contributions t', c', p', and p_1' in Table I. The amplitudes for $B^+ \to \eta K^+$ and $B^+ \to \eta' K^+$ for arbitrary mixing angles θ may be obtained using the definitions (3).

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Table I: Decomposition of $B^+ \to MM$ amplitudes for $|\Delta S| = 1$ transitions in terms of four independent quantities. Here η_p and η_p' denote the mixtures (5).

Final	Coefficient of			
state	t'	c'	p'	p_1'
$\pi^+ K^0$	0	0	1	0
$\pi^0 K^+$	$-1/\sqrt{2}$	$-1/\sqrt{2}$	$-1/\sqrt{2}$	0
$\eta_8 K^+$	$-1/\sqrt{6}$	$-1/\sqrt{6}$	$1/\sqrt{6}$	0
$\eta_1 K^+$	$1/\sqrt{3}$	$1/\sqrt{3}$	$2/\sqrt{3}$	$\sqrt{3}$
$\eta_p K^+$	$-1/\sqrt{3}$	$-1/\sqrt{3}$	0	$-1/\sqrt{3}$
$\eta_p' K^+$	$1/\sqrt{6}$	$1/\sqrt{6}$	$3/\sqrt{6}$	$2\sqrt{2/3}$

III. QUADRANGLE CONSTRUCTION

A. A specific example of $\eta - \eta'$ mixing

The amplitudes for $B^+ \to \pi^+ K^0$ contain only a contribution from p'. Since both the gluonic and electroweak penguin contributions of the amplitudes are dominated by the top quark, the weak phase of p' is $\pi = \mathrm{Arg}(V_{ts} V_{tb}^*)$, which does not change sign under charge conjugation. The same is true for the term p_1'. Hence we shall seek two linear combinations of amplitudes expressed in terms of p' and p_1'. When combined with $A(B^+ \to \pi^+ K^0)$, these will form an amplitude triangle whose shape will not change under charge conjugation.

The method can be illustrated using the special mixtures η_p and η_p' defined in Eq. (5), which are probably close to the physical states. We find

$$\sqrt{3} A(B^+ \to \eta_p K^+) + \sqrt{6} A(B^+ \to \eta_p' K^+) = 3(p' + p_1') \quad , \tag{10}$$

as well as

$$-\sqrt{2} A(B^+ \to \pi^0 K^+) + \sqrt{3} A(B^+ \to \eta_p K^+) = p' - p_1' \quad . \tag{11}$$

Combining these results with

$$A(B^+ \to \pi^+ K^0) = p' \quad , \tag{12}$$

we form a triangle with sides p', $p' - p_1'$, and $p' + p_1'$. This triangle will not change shape under charge conjugation.

For simplicity we define

$$a(\pi^+) \equiv A(B^+ \to \pi^+ K^0) \quad , \quad a(\pi^0) \equiv -\sqrt{2} A(B^+ \to \pi^0 K^+) \quad ,$$

$$a(\eta_p) \equiv \sqrt{3} A(B^+ \to \eta_p K^+) \quad , \quad a(\eta_p') \equiv \sqrt{6} A(B^+ \to \eta_p' K^+) \quad , \tag{13}$$
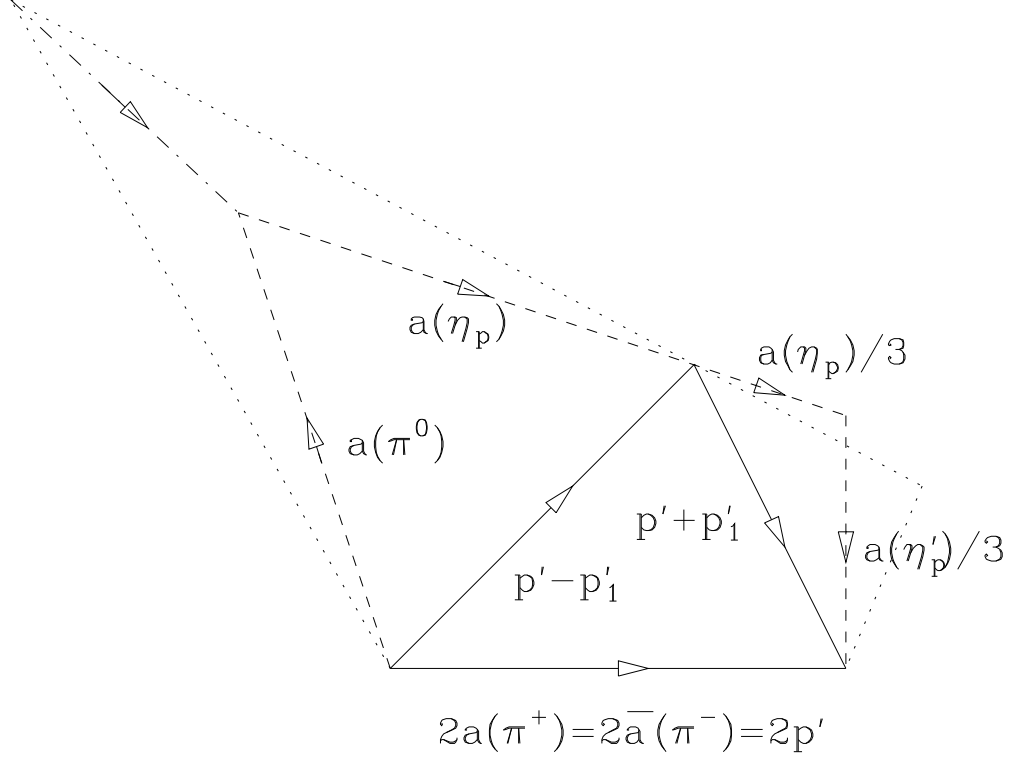
Figure 4: Quadrangle relation (15) satisfied by amplitudes (13) for B^+ decays (dashed lines, with solid line as base). The solid triangle corresponds to the linear relation among the combinations (14)). Dotted lines denote corresponding quadrangle for B^- decays. Here η_p and η'_p denote the octet-singlet mixtures (5). The dot-dashed line denotes $a(\pi^0) - \bar{a}(\pi^0) = t' + c' - (\bar{t}' + \bar{c}')$, whose magnitude and phase provide information on $\sin\gamma$ and a strong phase shift difference, respectively.

so that (10)–(12) may be transcribed as

$$\frac{1}{3}\left[a(\eta_p) + a(\eta'_p)\right] = p' + p'_1 \quad ,$$

$$a(\pi^0) + a(\eta_p) = p' - p'_1 \quad ,$$

$$a(\pi^+) = p' \quad , \tag{14}$$

and hence

$$\frac{4}{3}a(\eta_p) + \frac{1}{3}a(\eta'_p) + a(\pi^0) = 2a(\pi^+) \quad . \tag{15}$$

This quadrangle relation is illustrated in Fig. 4, along with the triangle formed by the three combinations in (14).

The charge-conjugate processes also satisfy relations equivalent to (14):

$$\frac{1}{3}\left[\bar{a}(\eta_p) + \bar{a}(\eta'_p)\right] = p' + p'_1 \quad ,$$

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$$\bar{a}(\pi^0) + \bar{a}(\eta_p) = p' - p'_1 \quad ,$$

$$\bar{a}(\pi^-) = p' \quad , \tag{16}$$

where

$$\bar{a}(\pi^-) \equiv A(B^- \to \pi^- \bar{K}^0) \quad , \quad \bar{a}(\pi^0) \equiv -\sqrt{2}A(B^- \to \pi^0 K^-) \quad ,$$

$$\bar{a}(\eta_p) \equiv \sqrt{3}A(B^- \to \eta_p K^-) \quad , \quad \bar{a}(\eta'_p) \equiv \sqrt{6}A(B^- \to \eta'_p K^-) \quad , \tag{17}$$

and hence

$$\frac{4}{3}\bar{a}(\eta_p) + \frac{1}{3}\bar{a}(\eta'_p) + \bar{a}(\pi^0) = 2\bar{a}(\pi^-) \quad . \tag{18}$$

The triangles formed by the combinations (14) and (16) are identical. Thus, the two quadrangles (15) and (18) must intersect at a point 3/4 of the distance from their upper left-hand vertices to their upper right-hand vertices. The shapes of the quadrangles are thus determined, up to discrete ambiguities which we shall discuss in Sec. III D and in the Appendix. This construction is reminiscent of one applied earlier to the decays $B \to \pi K$ and $\bar{B} \to \pi \bar{K}$ [18] in order to specify the shapes of amplitude quadrangles based on isospin.

Once the quadrangles are rigid, we can form the difference

$$a(\pi^0) - \bar{a}(\pi^0) = t' + c' - (\bar{t}' + \bar{c}') \quad , \tag{19}$$

where the bar denotes quantities appropriate to B^- decays. By an argument presented earlier [13], this difference can be utilized in conjunction with the amplitude for $B^\pm \to \pi^\pm \pi^0$ to extract both a strong phase difference and $\sin\gamma$. We shall recapitulate this argument in Sec. III C.

B. General mixing angle

The quadrangle relations are not much more complicated for a general mixing angle θ. We assume θ is measured by other means (see also Sec. IV B). The combinations corresponding to (14) are

$$\frac{\cos(\theta - \theta_0)}{\sqrt{3}}a(\eta) + \frac{\sin(\theta - \theta_0)}{\sqrt{6}}a(\eta') = p' + p'_1 \quad ,$$

$$\frac{a(\eta)}{\sqrt{3}} + \frac{\sin(\theta - \theta_0)}{\sqrt{2}}a(\pi^0) = p'\cos(\theta - \theta_0) - \sqrt{3}p'_1 \sin\theta \quad ,$$

$$a(\pi^+) = p' \quad . \tag{20}$$

Here $\theta_0 \equiv -\arcsin(1/\sqrt{3})$, the mixing angle for which η would be pure strange and η' would be pure nonstrange. We have retained the normalizations (13) for $a(\eta)$ and $a(\eta')$.

The quadrangle relation may be written as

$$a(\eta)\left[\sqrt{\frac{2}{3}}\csc(\theta - \theta_0) + \sqrt{2}\sin\theta\cot(\theta - \theta_0)\right] + a(\eta')\sin\theta + a(\pi^0) = 2a(\pi^+) \quad , \tag{21}$$
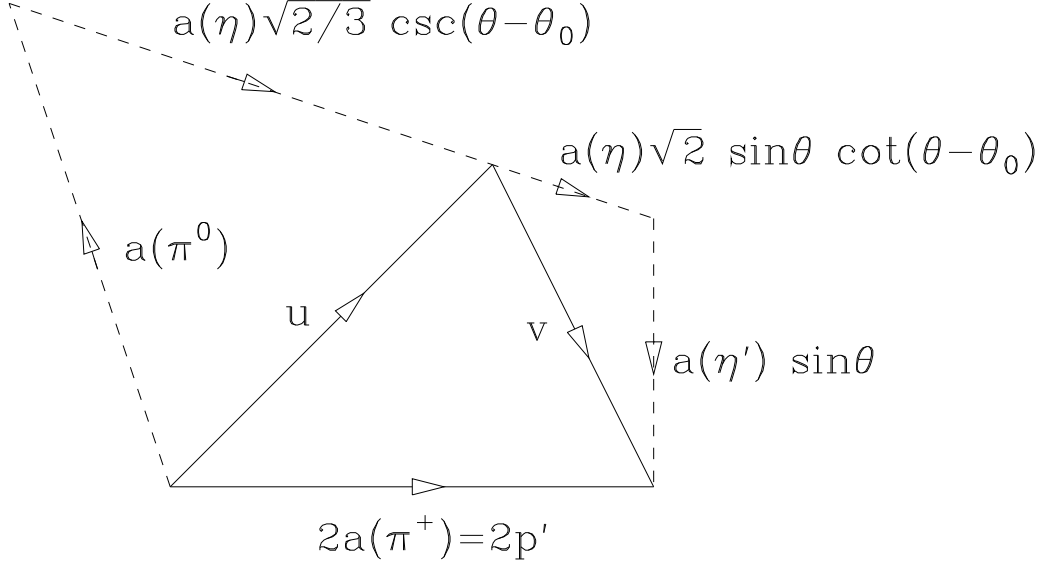
Figure 5: Quadrangle relation (21) satisfied by decay amplitudes (dashed lines, with solid line as base). The solid line corresponds to the linear relation among the combinations (20). The sum of the two $a(\eta)$ coefficients is $\sqrt{2}\cos\theta$.

as illustrated in Fig. 5. The two upper sides of the triangle invariant under charge conjugation are

$$u = \sqrt{2}[p'\cot(\theta - \theta_0) - \sqrt{3}p_1'\sin\theta\csc(\theta - \theta_0)] \quad ,$$

$$v = \sqrt{6}\sin\theta\csc(\theta - \theta_0)(p' + p_1') \quad . \tag{22}$$

We have shown the two coefficients of $a(\eta)$ in (21) separately in order to illustrate the construction of the invariant triangle. The two terms in the square bracket may be combined, leading to

$$\sqrt{2}a(\eta)\cos\theta + a(\eta')\sin\theta + a(\pi^0) = 2a(\pi^+) \quad , \tag{23}$$

which is just (2). The combination of η and η' decay amplitudes appearing in (23) is that corresponding to η_8, as also pointed out in Ref. [14].

Some special cases of (20), (23), and Fig. 5 may be noted.

When $\theta = 0$, the first two of Eqs. (20) reduce to

$$\frac{\sqrt{2}}{3}a(\eta_8) + \frac{1}{3\sqrt{2}}a(\eta_1) = p' + p_1' \quad ,$$

$$\frac{1}{\sqrt{3}}a(\eta_8) + \frac{1}{\sqrt{6}}a(\pi^0) = \sqrt{\frac{2}{3}}p' \quad . \tag{24}$$

The second equation may be combined with the last of (20) to obtain (1). This relation also follows directly from (23) in the limit $\theta \to 0$.

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When $\theta = \theta_p$, close to the physical value, we have $\cos(\theta - \theta_0) = \sqrt{1/3}$, $\sin(\theta - \theta_0) = \sqrt{2/3}$, so the combinations (20) reduce to (14), while (23) reduces to (15). The sides u and v of the triangle in Fig. 5 reduce to $u = p' - p_1'$ and $v = p' + p_1'$.

The construction of rigid quadrangles proceeds as in the example of Sec. III A. The $\bar{a}$ amplitudes obey a quadrangle relation similar to (21) or (23), with the same invariant triangle. Knowing θ, we can mark off a point a suitable distance along the $a(\eta)$ or $\bar{a}(\eta)$ side of each quadrangle, which must be common to the two triangles. As in Fig. 4, we can then determine the quantity (19).

An alternative construction is clearly possible in which an invariant triangle is constructed with one side composed of a linear combination of $a(\pi^0)$ and $a(\eta')$ instead of $a(\pi^0)$ and $a(\eta)$.

C. Determination of γ

The amplitudes $a(\pi^0)$ and $\bar{a}(\pi^0)$, which consist of terms containing tree and penguin weak phases, can be expressed as

$$a(\pi^0) = |a_{\pi K}^T| e^{i\gamma} e^{i\tilde{\delta}_T} - |a_{\pi K}^P| e^{i\tilde{\delta}_P} \quad ,$$

$$\bar{a}(\pi^0) = |a_{\pi K}^T| e^{-i\gamma} e^{i\tilde{\delta}_T} - |a_{\pi K}^P| e^{i\tilde{\delta}_P} \quad , \tag{25}$$

where all strong phases are written relative to that of p'. Here we have used the fact that the weak phase of the electroweak penguin is π. Taking the difference, we find

$$a(\pi^0) - \bar{a}(\pi^0) = 2i \sin\gamma \, e^{i\tilde{\delta}_T} |a_{\pi K}^T| \quad . \tag{26}$$

We now use flavor SU(3) to relate $|a_{\pi K}^T|$, corresponding to an $I = 3/2 \, \pi K$ amplitude [13], to the corresponding $I = 2 \, \pi\pi$ amplitude. [Both belong purely to the **27** of SU(3).] We find [7, 8, 13], since $\sqrt{2}A(B^+ \to \pi^+\pi^0) = -(T + C) - (P_{EW}^C + P_{EW})$ and

$$T'/T = C'/C = |V_{us}/V_{ud}|(f_K/f_\pi) \quad , \tag{27}$$

(neglecting the electroweak penguin contributions) that

$$|a_{\pi K}^T| = |V_{us}/V_{ud}|(f_K/f_\pi)\sqrt{2}|A(B^\pm \to \pi^\pm\pi^0)| \quad . \tag{28}$$

It does not matter whether we use the amplitude for $B^+ \to \pi^+\pi^0$ or $B^- \to \pi^-\pi^0$; negligible CP asymmetry is expected in the rates since electroweak penguin contributions should be small here [10, 11, 13].

D. Discrete ambiguities

In general, one expects $|p_1'/p'| \neq 0$ and $\mathrm{Arg}(p_1'/p') \neq 0$, so that there is a non-trivial invariant triangle. In that case, it will be very hard to avoid CP violation in rates for $B^\pm$ decays to at least one of the modes $\pi^0 K^\pm$, $\eta K^\pm$, or $\eta' K^\pm$ if the standard picture
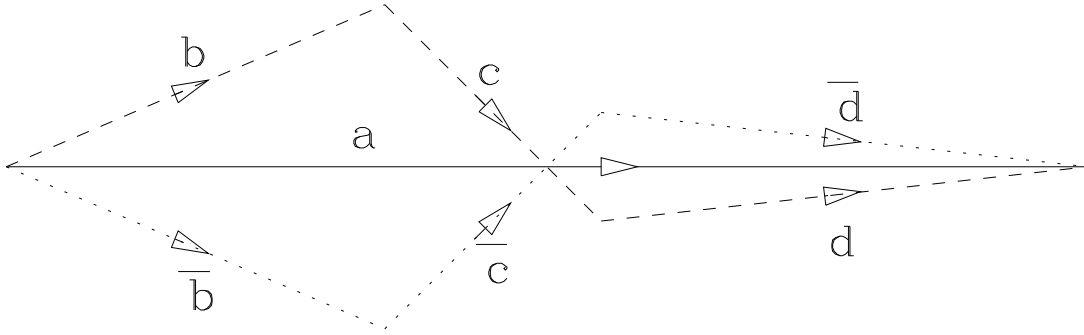
Figure 6: Example of degenerate quadrangles with no CP violation in $B^\pm$ decay modes. Here $a \equiv 2a(\pi^+) = 2\bar{a}(\pi^-)$, $b \equiv a(\pi^0)$, $c \equiv a(\eta)\sqrt{2}\cos\theta$, $d \equiv a(\eta')\sin\theta$, with barred quantities referring to corresponding B^- decay amplitudes.

with $\gamma \neq 0$ is correct. We shall argue in Sec. IV C that the $\eta K^\pm$ modes may hold the best prospect for such an asymmetry.

The two quadrangles can be degenerate, for example, if all four sides were equal in pairs, which would correspond to the absence of CP-violating asymmetries in decay rates. A limiting case is illustrated in Fig. 6. The strong phases of p' and p_1' are identical in this example; the invariant triangle has zero area.

The folded quadrangles in Fig. 6 illustrate a discrete ambiguity which will hold in general. The intersection of suitably chosen points along the $a(\eta)$ and $\bar{a}(\eta)$ sides of the quadrangles may be constructed either with the quadrangles as shown in Fig. 5, or folded as in Fig. 6. When CP violation is not present in rates, so that the sides of the quadrangles are equal in pairs, one solution will correspond to $\sin\gamma = 0$, while the other (as illustrated in Fig. 6) will give a value of $\sin\gamma$ which may be compared with the range inferred from analysis of CKM parameters based on other experiments (such as CP violation in neutral kaon decays). Solutions in which the difference (26) yields an unphysical value $|\sin\gamma| > 1$ may be rejected.

Another trivial ambiguity corresponds to flipping Figs. 4 and 5 about the horizontal axis. By reference to Eq. (26), one can see that this corresponds merely to the replacement $\tilde{\delta}_T \to \pi - \tilde{\delta}_T$, but no change in γ.

The most general class of discrete ambiguities is illustrated by the solution presented in the Appendix. There, one works backward from the known magnitudes of $a(\eta)$ and $\bar{a}(\eta)$ and assumes a variable relative phase ϕ between them. Then, for given $|a(\pi^0)|$, $|\bar{a}(\pi^0)|$, $|a(\eta')|$, and $|\bar{a}(\eta')|$, one finds a four-fold ambiguity in the solution for $|a(\pi^+)| = |\bar{a}(\pi^-)|$. The correct values of ϕ are those for which $|a(\pi^+)|$ equals the observed value.

IV. PRACTICAL CONSIDERATIONS

A. SU(3) breaking

The decays in question do not all have the same phase space. The correction factor $(\sim p_{\mathrm{cm}})$ is expected to be relatively small, since $p_{\mathrm{cm}}^{\pi K} = 2.61$ MeV/c, $p_{\mathrm{cm}}^{\eta K} = 2.59$ MeV/c,

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and $p_{\text{cm}}^{\eta' K} = 2.53$ MeV/c.

More important is the violation of SU(3) associated with the difference between creation of nonstrange and strange quark pairs in the final state in penguin (P') amplitudes. As discussed in a previous analysis [15], one does not really have a way to estimate this term. The creation of strange quark pairs occurs in the decays $B^+ \to \eta K^+$ and $B^+ \to \eta' K^+$, but not in $B^+ \to \pi K$ decays. For the mixed $\eta = \eta_p$ with $\theta = \theta_p = 19.5°$, exact SU(3) symmetry between nonstrange and strange quark pair creation would lead to the vanishing of the p' contribution (see Table I), which one might expect [13, 15] to be dominant. Assuming $|p_1'| < |p'|$, a suppression of the rate for $B^+ \to \eta K^+$ in comparison with the $B^+ \to \pi K$ and $\eta' K^+$ modes would be one piece of circumstantial evidence in favor of SU(3), though the contributions of the amplitudes besides p' would have to be taken into account as well.

B. Nature of η and η'

The representation of η and η' as quark-antiquark states with an octet-singlet mixing angle $\theta \simeq \theta_p$ is consistent with present data [16]. However, the admixture of non-$q\bar{q}$ states in the η and η' is less certain. By studying the decays $\rho \to \eta\gamma$ and $\phi \to \eta\gamma$, one can conclude that the non-$q\bar{q}$ admixture of the η is rather small [19]. The $u\bar{u} + d\bar{d}$ content of the η' is probed by the decay $\eta' \to \rho\gamma$, but one must await the measurement of the rate for $\phi \to \eta'\gamma$ to learn the $s\bar{s}$ content of the η' [19]. We have assumed that the gluonic content of η' is small.

C. Detection of final states

It is likely that the branching ratio for $B^0 \to \pi^- K^+$ is about 10^{-5} [20]. This process corresponds to an amplitude $A(B^0 \to \pi^- K^+) = -(t' + p')$, which should be dominated by the p' contribution. Thus $A(B^+ \to \pi^0 K^+) = -(t' + c' + p')/\sqrt{2}$, also expected to be dominated by p', should correspond to a branching ratio of about $(1/2) \times 10^{-5}$, while $A(B^+ \to \pi^+ K^0) = p'$ should correspond to a branching ratio of 10^{-5}. With a branching ratio $B(K^0 \to \pi^+\pi^-) \simeq 1/3$, the effective branching ratio for detecting $B^+ \to \pi^+ K^0$ via charged particles is $(1/3) \times 10^{-5}$.

The rate for the ηK^+ final state is harder to estimate. As we have mentioned, the p' contribution to this decay vanishes in the limit of exact SU(3) for a mixing angle of $\theta = \theta_p$, so that $A(B^+ \to \eta_p K^+) = -(t' + c' + p_1')/\sqrt{3}$. We estimated $|c'| \simeq |t'| \simeq (1/5)|p'|$ in Ref. [13], so that a branching ratio below 10^{-6} is a distinct possibility. (However, if the three terms t', c', and p_1' are of comparable magnitude and add constructively, the suppression of the rate for $B^+ \to \eta K^+$ could be relatively mild.) One must also take into account the efficiency whereby η's can be reconstructed, e.g. via their decays $\eta \to \gamma\gamma$ and $\eta \to \pi^+\pi^-\pi^0$.

One advantage of the $B^+ \to \eta K^+$ mode is that no one amplitude need be dominant, and there is room for differences in final-state phases (especially in comparing singlet and non-singlet amplitudes), so that this could well be a mode in which CP violation shows up as a difference in the branching ratios for $B^+ \to \eta K^+$ and $B^- \to \eta K^-$.

The $\eta' K^+$ amplitude has a coefficient of p' equal to $3/\sqrt{6}$ for $\theta = \theta_p$. Thus the branching ratio $B(B^+ \to \eta' K^+)$ could well exceed 10^{-5}. The η' would be detectable, for example, through its $\rho\gamma$ and $\eta\pi^+\pi^-$ modes.

V. CONCLUSIONS

We have shown that one can measure the weak phase $\gamma \equiv \mathrm{Arg}(V_{ub}^* V_{cb}/V_{us}^* V_{cs})$ by determining the relative rates for $B^+ \to \pi^0 K^+$, $B^+ \to \pi^+ K^0$, $B^+ \to \eta K^+$, $B^+ \to \eta' K^+$, and the corresponding charge-conjugate processes. The method is based on construction of amplitude quadrangles satisfying a constraint that specifies their shapes up to possible discrete ambiguities. The difference between the complex amplitudes for $B^+ \to \pi^0 K^+$ and $B^- \to \pi^0 K^-$, when compared with the magnitude of the amplitude for $B^\pm \to \pi^\pm \pi^0$, then yields both $\sin\gamma$ and some information on differences of strong-interaction phase shifts.

Key assumptions in this program include the validity of nonet symmetry for η and η' (testable to some extent in electromagnetic processes [5, 19]), the neglect of amplitudes scaling as f_B (testable in relations proposed earlier [7]), and the validity of SU(3) itself (testable to a limited extent in other cases [15] but only indirectly here through a possible suppression of the decays $B^\pm \to \eta K^\pm$). Indeed, we expect the detection of these decays to be the most demanding aspect of the present construction. Once the $\eta K^\pm$ mode has been seen, the major hurdle will have been overcome, with the possibility of a CP asymmetry in $B \to \eta K^\pm$ rates. Along the way, we expect the decays $B^\pm \to \eta' K^\pm$ to make a prominent appearance at the branching ratio level of 10^{-5} or greater.

APPENDIX: EXPLICIT CONSTRUCTION OF QUADRANGLES

We show in Fig. 7 the way in which the shapes of the quadrangles may be explicitly determined using only the lengths of the sides, the fact that they share a side, and the fact that the opposite sides intersect at a point corresponding to a fixed ratio of their lengths. The discussion is presented for simplicity for the case $\theta = \theta_p = \arcsin(1/3)$. Letters denote magnitudes of amplitudes, whereas in Fig. 6 they denoted the complex amplitudes.

We first construct lines with length $c = (4/3)|a(\eta)|$ and $\bar{c} = (4/3)|\bar{a}(\eta)|$ intersecting at a variable angle ϕ at a point $3/4$ of the distance along their lengths. We orient these lines so that the points C and $\bar{C}$ are joined by a line parallel to the x axis, and call the
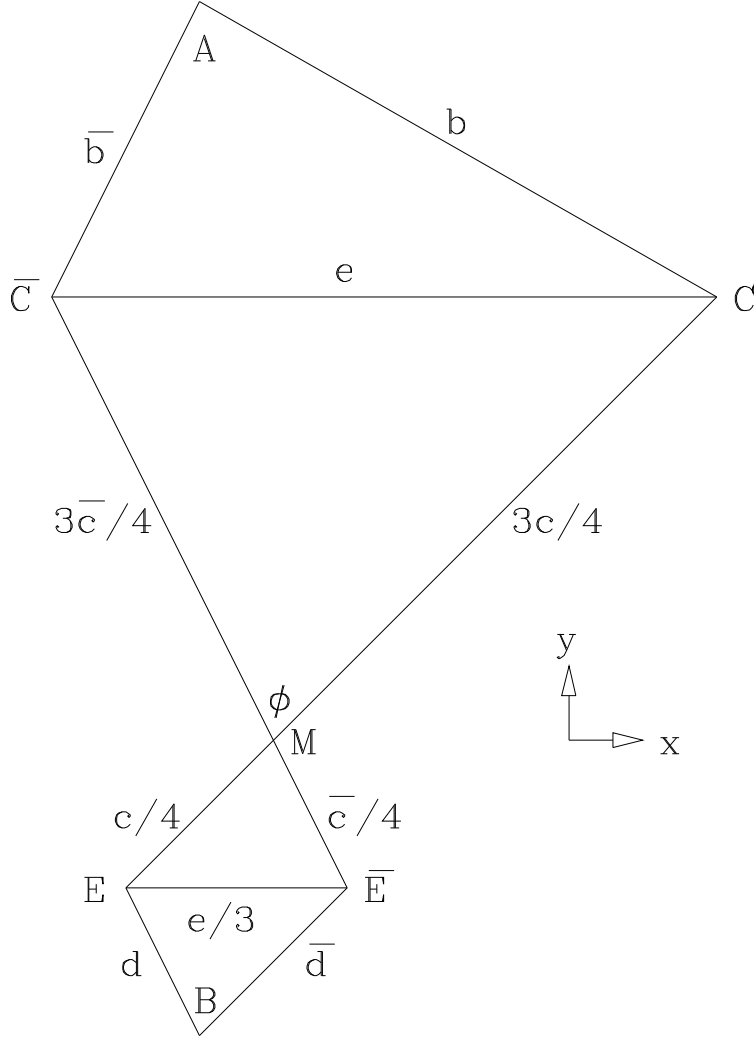
Figure 7: Explicit construction of amplitude quadrangles. Here we denote $|a(\pi^0)| \equiv b$, $|a(\pi^0) - \bar{a}(\pi^0)| \equiv e$, $|a(\eta)| \equiv 3c/4$, $|a(\eta')|/3 \equiv d$, and $2|a(\pi^+)| = 2|\bar{a}(\pi^-)| \equiv AB = a$, with corresponding notation for barred quantities.

intersection point M the origin. The length of the line $C\bar{C}$ will be $e = |a(\pi^0) - \bar{a}(\pi^0)|$; that of $E\bar{E}$ will be $e/3$ and $E\bar{E}$ will also be parallel to the x axis.

Now construct the triangle $C\bar{C}A$ with sides $b = |a(\pi^0)|$, $\bar{b} = |\bar{a}(\pi^0)|$, and e, and the triangle $E\bar{E}B$ with sides $d = |a(\eta')|/3$, $\bar{d} = |\bar{a}(\eta')|/3$, and $e/3$. The distance $a = AB$ must be given by $2|a(\pi^+)|$. This determines the angle ϕ. There is at most a four-fold ambiguity in this construction, corresponding to the possibility of flipping either triangle about its base.

It is easily seen that the above construction specifies the distance $a = AB$ up to a four-fold ambiguity, given any value of e for which the two lines CE and $\bar{C}\bar{E}$ can actually intersect. For example, the height h_A of the triangle $C\bar{C}A$ (its projection along the y axis) is

$$h_A = \frac{1}{2e}\left[-\lambda(b^2, \bar{b}^2, e^2)\right]^{1/2} \quad, \tag{29}$$

where $\lambda(a, b, c) \equiv a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$, with similar expressions for the heights g_A, $g_B = g_A/3$, and h_B of the respective triangles $C\bar{C}M$, $E\bar{E}M$, and $E\bar{E}B$. Then $y_A - y_B = g_A + g_B \pm h_A \pm h_B$ is specified up to the four-fold ambiguity mentioned above. The x coordinates of A and B may be obtained from expressions such as

$$x_C - x_A = \frac{1}{2e}\left[e^2 + b^2 - \bar{b}^2\right] \quad, \tag{30}$$

with similar expressions for x_C, x_E, and $x_B - x_E$. Thus one can also obtain $x_A - x_B$, and hence $AB = [(x_A - x_B)^2 + (y_A - y_B)^2]^{1/2}$.

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